JOURNAL OF COMPUTATIONAL PHYSICS 77, 40–52 (1988)

Adaptive Methods for Elliptic Grid Generation

KENICHI MATSUNO* AND HARRY A. DWYER

University of California, Davis, Davis, California

Received November 7, 1986; revised April 3, 1987

INTRODUCTION

The purpose of this research study is to develop a general method for the application of accurate and efficient adaptive gridding techniques to elliptic grid generation. Our goal is to develop a method that retains all the features of the robust solver pioneered by Thompson, Ref. [1], but which can incorporate a wide variety of adaptive strategies, such as first and second derivative function adaption. The method developed in the present paper is of this type, and it will be shown that it approximates high gradient functions very well, and extends to multi-dimensions in a direct fashion. In order to show the accuracy and usefulness of the techniques developed a series of examples will be presented in one and three space dimensions, which exhibit that the present methods can accurately approximate high gradient functions. These examples are independent of the efficiency and convergence of the transport equation algorithms, and offer a true test of adaptive gridding methods potential (although the positive or negative interaction of adaptive methods with an algorithm should not be overlooked). It will also be pointed out that grid adaption has its greatest benefits for problems where there is an asymptotic steepening of the dependent variable, such as shocks and flames. Smooth function variation is more efficiently treated by higher order methods such as finite element techniques.

The paper will begin with a derivation of the basic method of approach in one dimension, and it will be tested for accuracy and flexibility against a known asymptotic function. In many previous studies there has not been enough testing of the adaptive techniques for accuracy, and many investigators have only been concerned with the formulation of an adaptive method. In any real problem of function approximation there are many practical problems of normalization and free constant optimization which must be carried out to obtain good results. After the method has been shown to give promising results for the one-dimensional case, it will then be developed for three-dimensional applications and examples given. It is the present authors opinion that adaptive methods offer the hope of significant improvement in the future for numerical simulation of fluid flow and other phenomena that have asymptotic function variation in their solutions.

* Permanent address: National Aerospace Laboratories, Chofu, Tokyo, 182, Japan.

0021-9991/88 $3.00

Basic Method of Approach

The starting point for this new approach has been the Poisson equation for grid generation that has been well described in Ref. [1]. This equation for one space dimension is

$$\xi_{xx} = P(\xi), \tag{1}$$

where $P(\xi)$ is a source term which has been used in the past to impose such desired properties as grid expansion from a boundary surface or orthogonalization near a boundary surface. The above equation can be inverted to give a standard form, which is useful for solving for the unknowns x as a function of ξ and is given as

$$x_{\xi\xi} = -(P/J^2)\, x_\xi, \tag{2}$$

where J is the jacobian of the transformation and is given by

$$J = \xi_x.$$

The purpose of grid adaption in the present method is to find functions for P which adapt the grid on properties of the solution and still retain the known attributes of the above Poisson equation. In general, it is desirable to define a weighting function w, such that the physical change in x is small when w is large, or functionally

$$w x_\xi = C \text{ (constant)}. \tag{3}$$

The combination of Eqs. (2) and (3) yields

$$x_{\xi\xi} = -(w_\xi/w)\, x_\xi, \qquad \text{where} \quad P = J^2 w_\xi/w. \tag{4}$$

In order to be able to solve an equation such as (4) for realistic conditions there must be restrictions put on P which depend on the choice of w. These conditions will now be derived and sample solutions obtained.

If a central difference approximation is employed with Eq. (4), the resulting difference equation becomes

$$r_i - 1 = -(P/J^2)_i\,(r_i + 1)/2, \tag{5}$$

where $r_i = \Delta x_{i+1}/\Delta x_i$ and $\Delta x_i = x_i - x_{i-1}$.

The most basic constraint on the grid spacing ratio, r_i in Eq. (5), is that it must be positive for any value of i, and this leads directly to

$$|(P/J^2)_i| = |(w_\xi/w)_i| < 2. \tag{6}$$

A useful restriction is to place a limit on the maximum value of r_i, such as K or

$$1/K \leqslant r_i \leqslant K. \tag{7}$$

Combining Eqs. (6) and (7) yields the following constraint on the difference equation:

$$|(P/J^2)_i| = |(w_\xi/w)_i| \leqslant 2(K-1)/(K+1). \tag{8}$$

One of the most useful forms for the adaptive weighting function w is that form where it depends on the first derivative of the variable, f, to be adapted. First derivative adaption limits the percentage change of f and also forces the grid to be aligned perpendicularly to the gradient of f. Application of first derivative adaption to many different types of problems, Refs. [3–6], has shown that it is very accurate and can considerably increase the efficiency of the method of solution of the problem solved. A useful form of first derivative adaption is the total arclength of the variable f, and it can be written as

$$w = [1 + b_1(f_x)^2]^{1/2} \tag{9}$$

which leads directly from Eq. (8) to the value of b_1 as

$$(b_1)_i \leqslant [4(K-1)/(K+1)]/[|\{(f_x)^2\}_\xi| - 4(K-1)/(K+1)(f_x)^2]_i, \tag{10}$$

where the minimum value of $(b_1)_i$ has to be chosen. (That is $b_1 = \min[(b_1)_i]$.)

For the majority of problems occurring in the physical sciences first derivative adaption along with the restriction on the grid spacing ratio is adequate to obtain good adaption on the desired function. At the present time the numerical algorithms available are usually not robust enough to handle adaption on the second derivative of a dependent variable. The reason for this behavior is caused by the second derivative being poorly approximated, and adaption on faulty information can quickly lead to complete deterioration of the numerical algorithm. If a very robust algorithm is available for the solution of the equation for f, then second derivative adaption can be accomplished with the present methods. A useful formulation containing both the first and second derivatives of the function f is

$$w = w_1 w_2 = [1 + b_1(f_x)^2]^{1/2}\,[1 + b_2(f_{xx})^2]^{1/2}. \tag{11}$$

When this weighting function is utilized in the Poisson equation for the x positions, the results are

$$P/J^2 = w_\xi/w = w_{1\xi}/w_1 + w_{2\xi}/w_2$$

or, with the use of the grid spacing ratio restriction,

$$|P/J^2| = |(P_1/J^2 + P_2/J^2)_i| \leqslant 2(K-1)/(K+1).$$

The relative importance of first and second derivative adaption can be controlled by introducing the two constants K_1 (first derivative) and K_2 (second derivative) and the relative weighting parameter θ, which have the properties

$$K_1 + K_2 = 2(K-1)/(K+1),$$

where $K_1 = \theta[2(K-1)/(K+1)]$ and $K_2 = (1-\theta)[2(K-1)/(K+1)]$.

The values of b_1 and b_2 in w follow in the same way as the minimum values obtained from the equations

$$b_1 \leqslant 2K_1/[|\{(f_x)^2\}_\xi| - 2K_1(f_x)^2]_i \tag{12}$$

$$b_2 \leqslant 2K_2/[|\{(f_{xx})^2\}_\xi| - 2K_2(f_{xx})^2]_i. \tag{13}$$

Application of the One-Dimensional Method

In order to show the usefulness of the formulation just derived it will be applied to the "blind" approximation of a difficult function. The adjective "blind" has been employed, since the Poisson equation utilized to iteratively calculate the adapted values of the grid x, only knows discrete values of f (the same as would be returned by a finite difference equation). The function chosen is asymptotically steep and has radically different values of the first and second derivatives. This function is

$$f(x) = [1 + \text{sign}(x - x_c)\{1 - \exp(-aX^2 + 1/2)\}]/2, \tag{14}$$

where $X = 1/(2a)^{1/2} + |x - x_c|$, $a = 500$, $x_c = \frac{1}{2}$, and

$$\text{sign}(x - x_c) = 1 \qquad \text{if} \quad x \geqslant x_c$$

$$\text{sign}(x - x_c) = -1 \qquad \text{if} \quad x < x_c.$$

Shown in Figs. (1) through (4) are examples of grid adaption and function approximation for both first and second derivative weighting functions used for the

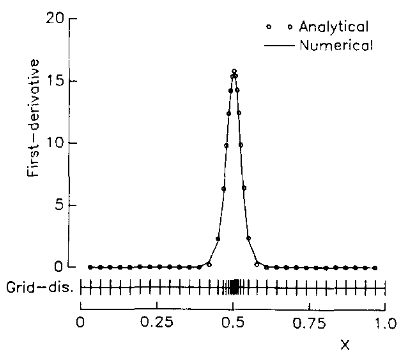

FIG. 1. Adaptive grid and first derivative. Approximation $\theta = 1$ and $K = 2$.

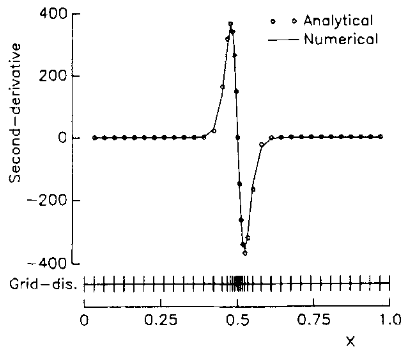

FIG. 2. Adaptive grid and second derivative. Approximation $\theta = 1$ and $K = 2$.

function w. Figures (1) and (2) exhibit the converged grid with only first derivative adaption for w ($\theta = 1$), and the exact and central difference approximations to both the first and second derivatives of the above asymptotic function. The comparison between these central difference approximations on this stretched grid, $K = 2$, and the analytic evaluations of the function are extremely good for the 41 grid points utilized. It should also be noted that the convergence to the grid shown only required 15 iterations of the Poisson equation to obtain the values shown in the figures. For every iteration the value of b_1 was updated, as well as the function. For values of a in equation (14) that were greater than 500 there was some difficulty with the convergence of the Poisson equation (it is even possible that the convergence of the grid is not obtained). In general, it can be said that it is more difficult to solve the Poisson equation for grid generation than to solve the integral equation employed in Ref. [4], and that better methods for obtaining the solution of the grid Poisson equation should be developed.

If the value of the weighting function, θ, between first and second derivative adaption is changed to one-half, the comparison between analytic and calculated results, Figs. (3) and (4), have the same general level of approximation, and it can

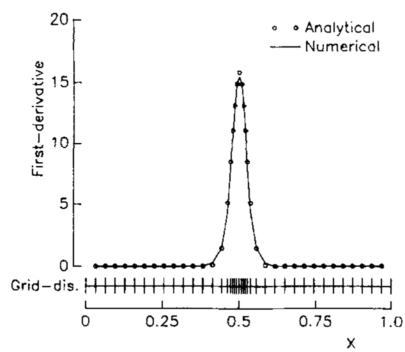

FIG. 3. Adaptive grid and first derivative. Approximation $\theta = \frac{1}{2}$ and $K = 2$.

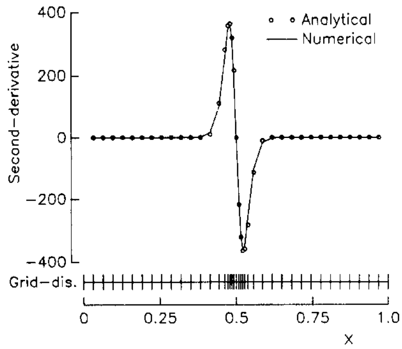

FIG. 4. Adaptive grid and second derivative. Approximation $\theta = \frac{1}{2}$ and $K = 2$.

be said that they seem to be of equal quality. This result exhibits the very common conclusion that first derivative adaption by itself usually does a very good job on the entire function, when it is combined with a reasonable finite difference approximation. This fact in conjunction with the realization that second derivative adaption usually causes convergence problems leads one to recommend the use of second derivative adaption only in very unusual circumstances. It should also be remembered that most finite difference approximations to partial differential equations are not nearly as robust as knowing the function exactly, and do not interact as well with grid motion.

The use of adaptive methods with the present formulation does not preclude the use of purely geometric control terms in the Poisson equation for grid generation. For example, if it is required to have a geometrically expanding grid (this type of grid variation is usually applied to resolve boundary layers), as well as grid adaption, then the total control function P becomes

$$|P/J^2| = |(P_a + P_g)/J^2| \leqslant 2(K-1)/(K+1), \tag{15}$$

where $P_g = -2(r_g - 1)/(r_g + 1)$ for $r_g \geqslant 1$.

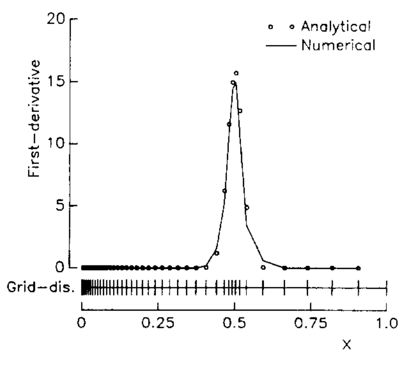

FIG. 5. Adaptive grid and first derivative. Approximation $\theta = 1$, $K = 4$, and $r_g = 1.075$.

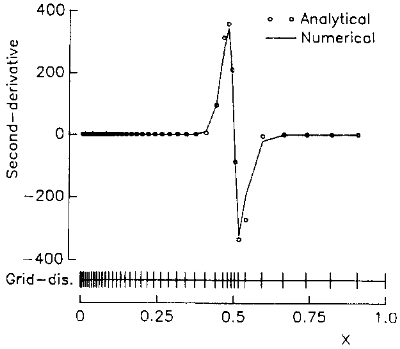

FIG. 6. Adaptive grid and second derivative. Approximation $\theta = 1$, $K = 4$, and $r_g = 1.075$.

When this formulation is used with first derivative adaption the results again are very good with the same number of total grid points (Figs. 5, 6). Of course, if more grid points were added, it would be possible to improve the results to the levels of Figs. 1–4. In general, additional grids are needed for each high gradient region in a calculation, but it is seen from the above results that many different types of high gradient regions can be nicely resolved with the present methods.

THREE-DIMENSIONAL FORMULATION

A very attractive feature of the above formulation is that it can readily be extended to a multi-dimensional form. The form which has been chosen for the present results has been to adapt along the three physical arclength directions defined by the generalized coordinates ξ, η, and ζ, Fig. 7. The three-dimensional form of the Poisson equations are

$$\xi_{xx} + \xi_{yy} + \xi_{zz} = P(\xi, \eta, \zeta)$$

$$\eta_{xx} + \eta_{yy} + \eta_{zz} = Q(\xi, \eta, \zeta)$$

$$\zeta_{xx} + \zeta_{yy} + \zeta_{zz} = R(\xi, \eta, \zeta),$$

where $P = w_\xi/[(s_\xi)^2 w]$, $Q = w'_\eta/[(s'_\eta)^2 w']$, and $R = w''_\zeta/[(s''_\zeta)^2 w'']$.

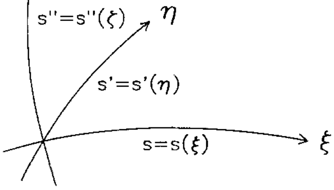

FIG. 7. Primary arclengths and generalized coordinates.

These equations can be inverted to give forms which can be used for numerical solution to obtain the grid

$$D'x = -(Px_\xi + Qx_\eta + Rx_\zeta)/J^2$$
$$D'y = -(Py_\xi + Qy_\eta + Ry_\zeta)/J^2$$
$$D'z = -(Pz_\xi + Qz_\eta + Rz_\zeta)/J^2,$$

where the differential operator D' is defined as

$$D'x = \alpha x_{\xi\xi} + \beta x_{\eta\eta} + \gamma x_{\zeta\zeta} + 2\kappa x_{\xi\eta} + 2\lambda x_{\eta\zeta} + 2\mu x_{\zeta\xi}$$

and the coefficients α, β, γ, κ, λ, μ, and J are functions of the coordinate transformation.

For those readers who are familiar with the excellent research efforts based on a variational principle, Refs. [1, 6, 7], the present formulation may seem to be somewhat different, but in reality it is almost identical to that proposed by Winslow [6]. If the gradient operator, which acts on the adaption variable w in the Winslow formulation, is expanded in the generalized coordinates ξ, η, and ζ, it can be shown that the formulations are almost identical. For example, $w_\xi \xi_x$ in the Winslow formulation can be shown to be equal to $w_\xi x_\xi / s_\xi^2$ in the present work with the use of some calculus and the definition of the arclength variable (similar results can be obtained for all nine source terms in the equations for x, y, and z given above). For the formulations to be identical the adaption functions w, w', and w'' must be the same along all three arclengths, but this possible flexibility in w exhibits clearly the use of different adaption functions along different physical directions. In many applications the dominant physics is strongly modified by boundaries or external force fields, and there are definite advantages to changing the adaption function along the three independent directions.

The weighting functions w, w', and w'' contain the information on the adaption criteria to be used in a given problem, and typical first and second derivative formulations along the three arclength directions are

$$w = [1 + b_1(F_s)^2]^{1/2} [1 + b_2(F_{ss})^2]^{1/2}$$
$$w' = [1 + b_1'(F_{s'})^2]^{1/2} [1 + b_2'(F_{s's'})^2]^{1/2}$$
$$w'' = [1 + b_1''(F_{s''})^2]^{1/2} [1 + b_2''(F_{s''s''})^2]^{1/2}.$$

In general, all of the above comments on the usefulness of first and second derivative adaption apply to the three-dimensional case, but with an increased sensitivity due to the difficulties with most three-dimensional algorithms. The additional question of grid orthogonality also must be addressed for the multi-dimensional cases, and this places a significant restriction on adaption. As has been shown in a previous paper, Ref. [6], orthogonality and grid adaption cannot be strictly applied simultaneously, and they must be traded off against each other.

However, orthogonality is a geometric constraint, and can be best handled by the introduction of a weighting function P_G, which will compete with the adaption source term. In most problems it is best that adaption be applied along a coordinate direction for which diffusion or an elliptic variable is important, and orthogonality emphasized along the perpendicular directions.

The constants b_1 and b_2 in the weighting functions for the first and second derivatives, respectively, are determined in a fashion similar to the one-dimensional case. If K is the overall grid spacing ratio, and θ the parameter determining the relative importance of first and second derivative adaption, the restrictions on b_1 and b_2 become

$$b_1 \leqslant 2K_1/[|(F_s)_\xi^2| - 2K_1(F_s)^2]$$

$$b_2 \leqslant 2K_2/[|(F_{ss})_\xi^2| - 2K_2(F_{ss})^2]$$

with $K_1 + K_2 = \theta[2(K-1)/(K+1)] + (1-\theta)[2(K-1)/(K+1)]$.

Of course the optimization parameters in the other coordinate directions are free to be chosen independently and should be chosen independently in order to use the grid more efficiently.

Three-Dimensional Applications

The test problem chosen to evaluate the present formulation for the three-dimensional case is similar to the one-dimensional problem previously described. The asymptotic function used for adaption is given as

$$F(x, y, z) = [1 + \text{sign}(r - r_c)\{1 - \exp(-a\rho^2 + 1/2)\}]/2,$$

where $r = (x^2 + y^2 + z^2)^{1/2} \leqslant 1$ and $\rho = 1/(2a)^{1/2} + |r - r_c|$, $a = 500$, and $r_c = 1/2$.

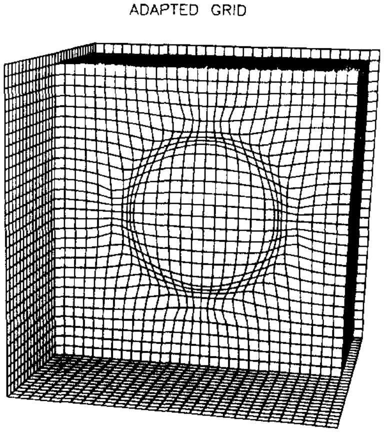

FIG. 8. Three-dimensional adaptive grid.

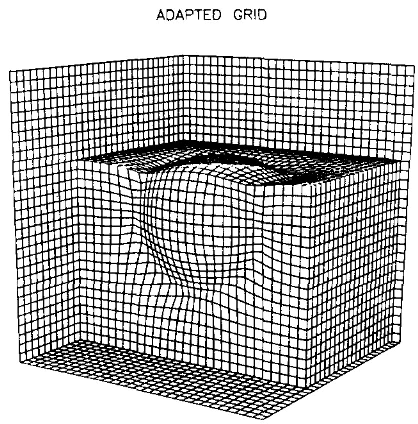

FIG. 9. Three-dimensional adaptive grid.

The adapted grid that resulted from the same "blind" numerical experiment is shown in Figs. 8–10 for three different perspective views. The initial grid for the start of adaption was a thirty-one cube uniform grid, which was provided with discrete values of the function F. Based on these values of the function F, the numerical first and second derivatives were formed and the Poisson equations solved with the use of Jacobi line relaxation. The converged solution required 38 total iterations, and it can be seen from the figures that the symmetry of the function and the grid are well matched.

The comparison with the first and second derivatives is almost as good as the one-dimensional grid with one-half the number of grid points, but it does suffer

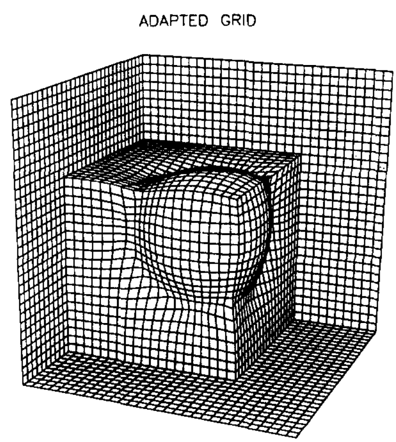

FIG. 10. Three-dimensional adaptive grid.

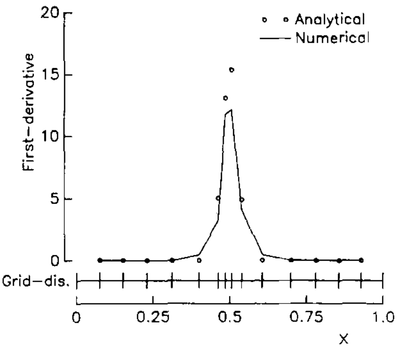

FIG. 11. Adaptive grid and second derivative. Approximation $\theta = 1$ and $K = 2$ for a 3D case.

slightly from the nonorthogonality of the grid (the reason for one-half the number of grid points comparison is due to the fact that the spherical function utilized causes two high gradient regions along any axis in space). A typical comparison of the first and second derivatives along the x-axis shown in Figs. 11 and 12. When the difficulty of the function and the sparse number of grid points is considered, it is actually quite promising for the use of adaptive gridding techniques. In general the use of nonorthogonal grids does degrade present central difference approximations in viscous flow regions, but the benefits of adaption more than offset these losses. In the near future there does seem to be good hope of increased storage and speed advances which offer the possibility of overcoming most of the present problems. It is the present authors' opinion that the adaptive method presented in this paper will be a useful and efficient way of helping to solve the difficult and ever more complex problems of the future.

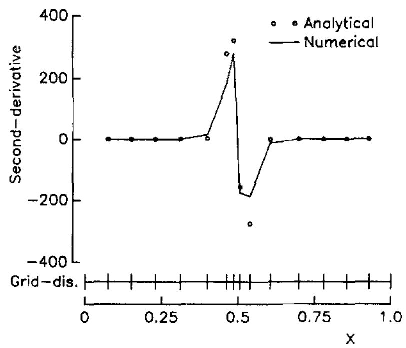

FIG. 12. Adaptive grid and second derivative. Approximation $\theta = 1$ and $K = 2$ for a 3D case.

CONCLUSIONS

The present paper has formulated a general elliptic solver method and calculated a series of examples for adapted grids on an asymptotic function which has very different first and second derivatives. The results of the study have been very promising and the following conclusions can be made:

1. The use of Poisson equations with source terms to adapt the grid on various properties of the dependent variable of a transport equation can be formulated for multi-dimensional grids. However, there are restrictions which must be applied to force the grid to have a positive grid spacing ratio. If these restrictions are not observed the use of elliptic grid generation will produce negative Jacobians.

2. Both first and second derivative adaption can be accomplished in a similar fashion with the use of appropriate weighting functions. The use of first derivative adaption is the most robust and useful form, and it also gives the major share of the accuracy for a function approximation. The use of second derivative adaption improves the approximation of the function but usually influences the convergence of the adaptive grid equation and/or the solution algorithm in a negative way.

3. The use of source terms in the Poisson equations to adapt the grid does not preclude the use of geometric source terms to give the grid desired properties near solid boundaries or expansions to a freestream condition. In fact, one of the major advantages of the present adaptive method is that it is totally compatible with the previous geometric methods of grid generation.

4. With the use of exact test functions to generate adaptive grids the convergence of the line Jacobi method for the solution of the Poisson equations was excellent, even for the three-dimensional problems solved. However, if the numerical algorithm employed to solve the transport equation is marginal, then there can be a negative interaction between the numerical algorithm and grid adaption.

5. The major problem to be treated in the future is the nonorthogonal and badly skewed grids that are generated by function variation and complex geometry. This problem can be severe since orthogonality and adaption cannot be applied simultaneously. In many physical problems this interaction is reduced since diffusion and convection are perpendicular to each other at high Reynolds number.

REFERENCES

1. J. F. THOMPSON, Z. U. A. WARSI, AND C. W. MASTIN, *Numerical Grid Generation* (Elsevier Science, New York, 1985).
2. R. L. SORENSON AND J. L. STEGER, *in Proceedings, ASME Mechanics, Bioengineering, and Fluids Engineering Conference, Houston, 1983*, FED Vol. 5, edited by K. N. Ghia.

3. K. Nakahashi and G. S. Deiwert, "A Self-Adaptive-Grid Method with Application to Airfoil Flow," AIAA 7th Comput. Fluid Dynamics Conf., Cinn., Ohio, July, 1985.
4. H. A. Dwyer and O. Onikza, in *Proceedings, 9th Inter. Conf. on Numerical Methods in Fluid Mechanics, Paris, France, July 1984*, Lecture Notes in Physics (Springer-Verlag, Berlin/New York, 1985).
5. H. A. Dwyer, Lectures for Reacting Flows, Cornell University, May 1985; *Physica* **20D**, 142 (1986).
6. A. M. Winslow, UCID-19062, Lawrence Livermore National Laboratory, 1981 (unpublished).
7. J. U. Brackbill, Solution adaptive grids for time-dependent problems in two dimensions, Division of Applied Mathematics, Los Alamos National Laboratories (unpublished).